Dynamical Correlations among Vicious Random Walkers

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Nonintersecting motion of Brownian particles in one dimension is studied. The system is constructed as the diffusion scaling limit of Fisher's vicious random walk. N particles start from the origin at time $t = 0$ and then undergo mutually avoiding Brownian motion until a finite time $t = T$. In the short time limit $t \ll T$, the particle distribution is asymptotically described by Gaussian Unitary Ensemble (GUE) of random matrices. At the end time $t = T$, it is identical to that of Gaussian Orthogonal Ensemble (GOE). The Brownian motion is generally described by the dynamical correlations among particles at many times $t_1, t_2, \cdots, t_M$ between $t = 0$ and $t = T$. We show that the most general dynamical correlations among arbitrary number of particles at arbitrary number of times are written in the forms of quaternion determinants. Asymptotic forms of the correlations in the limit $N \to \infty$ are evaluated and a discontinuous transition of the universality class from GUE to GOE is observed.

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The vicious walk model in which many walkers randomly move without intersecting with others was introduced by Fisher and applied to wetting and melting phenomena [1]. Recently it has attained a renewed interest, since intimate relations to other research fields, such as the theory of Young tableaux in combinatorics [2–4], asymmetric exclusion process (ASEP) in nonequilibrium statistical mechanics [5], polynuclear growth (PNG) model of surface physics [6,7] and the theory of random matrices [8,9], have been revealed one after another and brought progress in the study of these topics.

One of the reigning concepts of these new applications, the universality class, comes from the theory of random matrices. Gaussian Orthogonal Ensemble (GOE) and Gaussian Unitary Ensemble (GUE) universality classes originate in the real symmetric and complex hermitian structures of random matrices. They also appear in the new applications in quite natural ways, despite the absence of underlying matrix structure. In the theory of Young tableaux, number permutation and involution correspond to GUE and GOE, respectively [10]. In ASEP, the initial and boundary conditions demarcate the universality classes of the current fluctuation [11,12]. The height fluctuation of surface growth on a flat substrate belongs to the GOE class and a droplet growth is described by GUE [7].

How the universality classes appear in the vicious walk model? The GOE universality class is realized when walkers take t steps under the nonintersecting condition. One possibility to observe the GUE class is to impose an additional condition that the nonintersecting walkers come back to the original position after $2t$ steps [8,9,11]. In analyzing the diffusion scaling limit of the vicious walk model, Katori and Tanemura [13,14] recently noticed another possibility. The walker distribution depends not only on the observation time (the number of steps) t but also on the time interval T in which the nonintersecting condition is imposed. Suppose that all the walkers start from the origin at time $t = 0$. While the ratio of the time t and the nonintersection time interval T, t/T, is small, the walker distribution is asymptotically described by GUE. When the end of the nonintersection time interval arrives, namely at $t/T = 1$, the distribution becomes identical to that of GOE. This means that, while walkers randomly move in the time interval between $t = 0$ and $t = T$, a transition from GUE to GOE takes place.

In this Letter we analyze the dynamical correlations among the vicious walkers in the transition region. Utilizing the equivalence to a multimatrix model in quantum field theory, we evaluate the dynamical correlation functions among arbitrary number of walkers at arbitrary number of times. The asymptotic limit of the large number of walkers will be evaluated and we will find that the transition becomes discontinuous.

Let us consider N independent symmetric simple random walks on $\mathbf{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ started from N distinct positions $2s_1 < 2s_2 < \cdots < 2s_N$, $s_j \in \mathbf{Z}$. The position of the j-th random walker at time $k \geq 0$ is denoted by $R_k^{s_j}$ and we impose the nonintersecting condition

$$R_k^{s_1} < R_k^{s_2} < \cdots < R_k^{s_N}, \quad 1 \leq \forall k \leq K. \tag{1}$$

If a possible random walk satisfies the condition (1), then it is called a vicious walk. Let $V(R_K^{s_j} = e_j)$ be the realization probability of the vicious walks, in which the N walkers arrive at the positions $2e_1 < 2e_2 < \cdots < 2e_N$, $e_j \in \mathbf{Z}$,

at time K.

A simplification is obtained in the diffusion scaling limit: we set $K = Lt, s_j = \sqrt{L}x_j/2, e_j = \sqrt{L}y_j/2$ and take the limit $L \to \infty$. In Ref. [13], it is clarified that $\lim_{L\to\infty}(\sqrt{L}/2)^N V(R_{Lt}^{\sqrt{L}x_j/2} = \sqrt{L}y_j/2) = f(t;\{y_j\} \mid \{x_j\})$, where

$$f(t;\{y_j\} \mid \{x_j\}) = \frac{1}{(2\pi t)^{N/2}} \det[e^{-(x_j-y_k)^2/2t}]_{j,k=1,\cdots,N}. \tag{2}$$

This function shows how the nonintersecting probability of the Brownian particles on the rescaled lattice $\mathbf{Z}/(\sqrt{L}/2)$ up to time t depends on the initial and final positions. Therefore, for the nonintersecting Brownian motions in the time interval $[0,T]$, the transition density from the configuration $\{x_j\}$ at time s to $\{y_j\}$ at time t is given by [14]

$$\varphi^T(s,\{x_j\};t,\{y_j\}) = \frac{f(t-s;\{y_j\} \mid \{x_j\})\mathcal{N}(T-t;\{y_j\})}{\mathcal{N}(T-s;\{x_j\})}, \tag{3}$$

where $\mathcal{N}(t;\{x_j\}) = \int_{y_1<\cdots<y_N} \prod_{j=1}^N dy_j f(t;\{y_j\} \mid \{x_j\})$. Note that there is an obvious temporal inhomogeneity since RHS depends not only $t-s$ but also $T-s$ and $T-t$.

The dynamics of the Brownian particles is described by the dynamical correlation functions. Let us denote the positions of the Brownian particles at a time t_j by $x_1^j, x_2^j, \cdots, x_N^j$. Then the dynamical correlation functions among the particles at times $t_1, t_2, \cdots, t_M$ are defined as

$$\begin{aligned}
&\rho(x_1^1,\cdots,x_{n_1}^1;\cdots;x_1^M,\cdots,x_{n_M}^M) \\
&= \int \prod_{j=1}^N dx_j^0 \int \prod_{j=n_1+1}^N dx_j^1 \cdots \int \prod_{j=n_M+1}^N dx_j^M \\
&\times p_0(\{x_j^0\}) \prod_{m=0}^{M-1} \varphi^T(t_m,\{x_j^m\};t_{m+1},\{x_j^{m+1}\}).
\end{aligned} \tag{4}$$

Here $p_0(\{x_j^0\})$ is the initial distribution at $t_0 = 0$. Let us now suppose that all the particles start at the origin. Namely, we set $p_0(\{x_j^0\}) = \prod_j \delta(x_j^0)$ and obtain

$$\begin{aligned}
&\rho(x_1^1,\cdots,x_{n_1}^1;\cdots;x_1^M,\cdots,x_{n_M}^M) \\
&\propto \int \prod_{j=n_1+1}^N dx_j^1 \cdots \int \prod_{j=n_M+1}^N dx_j^M \int \prod_{j=1}^N dx_j^{M+1} \\
&\times \prod_{j>k}^N (x_j^1 - x_k^1) \prod_{j>k}^N \mathrm{sgn}(x_j^{M+1} - x_k^{M+1}) \\
&\times \prod_{m=1}^M \det[g^m(x_j^m, x_k^{m+1})]_{j,k=1,\cdots,N}.
\end{aligned} \tag{5}$$

Here ($t_{M+1} \equiv T$)

$$\begin{aligned}
g^1(x,y) &= \frac{e^{-x^2/(2t_1)} e^{-(x-y)^2/(2(t_2-t_1))}}{\sqrt{2\pi t_1}\sqrt{2\pi(t_2-t_1)}}, \\
g^m(x,y) &= \frac{e^{-(x-y)^2/(2(t_{m+1}-t_m))}}{\sqrt{2\pi(t_{m+1}-t_m)}}, \quad 2 \le m \le M.
\end{aligned} \tag{6}$$

At this stage we notice a direct correspondence between our problem and the multimatrix models in quantum field theory. Itzykson and Zuber firstly analyzed a two matrix model in which two hermitian random matrices were combined [15,16]. Mehta and Pandey then coupled a real-symmetric and a hermitian random matrices to devise a $(1+1)$ matrix model, as a mathematical interpolation of GOE and GUE [17,18]. After it had been realized that multimatrix models were useful in the quantum field theory on random surfaces [19], Eynard and Mehta invented a method to evaluate the correlation functions among the eigenvalues of combined M hermitian matrices [20]. As a further generalization, Nagao proposed an $(M+1)$ matrix model in which one real symmetric and M hermitian matrices were combined and showed that the correlation functions were generally written in the forms of quaternion determinants [21]. We can readily see that the above dynamical correlation functions among vicious walkers have the same forms as the eigenvalue correlation functions of the $(M+1)$ matrix model.

For simplicity we set N even and summarize the quaternion determinant formulas in the following. In terms of ($x_m \equiv x, x_n \equiv y$)

$$\begin{aligned}
&G^{mn}(x,y) \\
&= \begin{cases} \delta(x-y), & m=n, \\ g^m(x,y), & m=n-1, \\ \displaystyle\int \prod_{j=m+1}^{n-1} dx_j \prod_{l=m}^{n-1} g^l(x_l,x_{l+1}), & m<n-1, \end{cases}
\end{aligned} \tag{7}$$

we firstly define

$$\begin{aligned}
&F^{mn}(x,y) \\
&= \int_{-\infty}^{\infty} dz' \int_{-\infty}^{z'} dz \{G^{m\ M+1}(x,z) G^{n\ M+1}(y,z') \\
&- G^{n\ M+1}(y,z) G^{m\ M+1}(x,z')\}.
\end{aligned} \tag{8}$$

Let us introduce an antisymmetric inner product

$$\begin{aligned}
&\langle f(x), g(y) \rangle \\
&= \frac{1}{2} \int dx \int dy F^{11}(y,x)[f(y)g(x) - f(x)g(y)]
\end{aligned} \tag{9}$$

and construct monic polynomials $R_k^1(x) = x^k + \cdots$ of degrees k so that they satisfy the skew orthogonality relations:

$$\begin{aligned}
&\langle R_{2j}^1(x), R_{2l+1}^1(y) \rangle = -\langle R_{2l+1}^1(x), R_{2j}^1(y) \rangle = r_j \delta_{jl}, \\
&\langle R_{2j}^1(x), R_{2l}^1(y) \rangle = 0, \ \langle R_{2j+1}^1(x), R_{2l+1}^1(y) \rangle = 0.
\end{aligned} \tag{10}$$

We then define functions $R_k^m(x)$ and $\Phi_k^m(x)$, $m = 2,3,\cdots,M+1$, as

$$\begin{aligned} R_k^m(x) &= \int dy R_k^1(y) G^{1m}(y,x), \\ \Phi_k^m(x) &= \int dy R_k^m(y) F^{mm}(y,x). \end{aligned} \tag{11}$$

Now matrices D^{mn}, I^{mn} and S^{mn} are introduced as

$$D_{jl}^{mn} = \sum_{k=0}^{(N/2)-1} \frac{R_{2k}^m(x_j^m)R_{2k+1}^n(x_l^n) - R_{2k+1}^m(x_j^m)R_{2k}^n(x_l^n)}{r_k},$$

$$I_{jl}^{mn} = \sum_{k=0}^{(N/2)-1} \frac{\Phi_{2k+1}^m(x_j^m)\Phi_{2k}^n(x_l^n) - \Phi_{2k}^m(x_j^m)\Phi_{2k+1}^n(x_l^n)}{r_k},$$

and

$$S_{jl}^{mn} = \sum_{k=0}^{(N/2)-1} \frac{\Phi_{2k}^m(x_j^m)R_{2k+1}^n(x_l^n) - \Phi_{2k+1}^m(x_j^m)R_{2k}^n(x_l^n)}{r_k}.$$

Further we define

$$\tilde{S}_{jl}^{mn} = \begin{cases} S_{jl}^{mn}, & m \geq n, \\ S_{jl}^{mn} - G^{mn}(x_j^m, x_l^n), & m < n \end{cases}$$

and

$$\tilde{I}_{jl}^{mn} = I_{jl}^{mn} + F^{mn}(x_j^m, x_l^n).$$

For a self dual $N \times N$ quaternion matrix $Q = [q_{jk}]$, a determinant $\mathrm{Tdet}Q$, originally introduced into random matrix theory by Dyson [22], is defined as

$$\mathrm{Tdet}\, Q = \sum_P (-1)^{N-l} \prod_1^l \mathrm{tr}(q_{ab}q_{bc}\cdots q_{da}), \tag{12}$$

where P denotes any permutation of the indices $(1,2,\cdots,N)$ consisting of l exclusive cycles of the form $(a \to b \to c \to \cdots \to d \to a)$ and $(-1)^{N-l}$ is the parity of P. Note that $\mathrm{tr}q$ is equal to a half of the trace of the 2×2 matrix representation of q.

Let us suppose that the elements of quaternion matrices $B^{\mu\nu}$, $\mu,\nu = 1,2,\cdots,M$, have the following 2×2 representations:

$$B_{jl}^{\mu\nu} = \begin{bmatrix} \tilde{S}_{jl}^{\mu\nu} & \tilde{I}_{jl}^{\mu\nu} \\ D_{jl}^{\mu\nu} & \tilde{S}_{lj}^{\nu\mu} \end{bmatrix}, \quad j,l = 1,2,\cdots,N. \tag{13}$$

Then Nagao's result [21] asserts that

$$\begin{aligned} &\rho(x_1^1,\cdots,x_{n_1}^1;\cdots;x_1^M,\cdots,x_{n_M}^M) \\ &\propto \mathrm{Tdet}[B^{\mu\nu}(n_\mu,n_\nu)], \quad \mu,\nu = 1,2,\cdots,M, \end{aligned} \tag{14}$$

where each block $B^{\mu\nu}(n_\mu,n_\nu)$ is obtained by removing the $n_\mu+1, n_\mu+2,\cdots,N$-th rows and $n_\nu+1,n_\nu+2,\cdots,N$-th columns from $B^{\mu\nu}$.

Let us examine the consequences of the quaternion determinant formula. Now we remark that the skew orthogonal polynomials $R_k^1(x)$ are explicitly written as

$$R_k^1(x) = \xi^{k/2} \sum_{j=0}^k \alpha_{kj} H_j\left(\frac{x}{c_1}\right) \xi^{-j/2}, \tag{15}$$

where $\xi = t_1/(2T - t_1)$,

$$c_n = \sqrt{t_n(2T-t_n)/T},$$

$$\begin{aligned} \alpha_{2k\ j} &= 2^{-2k} c_1^{2k} \delta_{2k\ j}, \\ \alpha_{2k+1\ j} &= 2^{-2k-1} c_1^{2k+1} (\delta_{2k+1\ j} - 4k\delta_{2k-1\ j}) \end{aligned}$$

and $H_j(x)$ are the Hermite polynomials.

We begin with the simplest case $M = 1$ and $n_1 = 1$. Putting the explicit formula of $R_k^1(x)$ into the quaternion determinant expression and utilizing the asymptotic formula for the Hermite polynomials, we can readily obtain

$$\rho(x_1^1) \propto \frac{1}{\pi c_1}\sqrt{2N - (x_1^1/c_1)^2}, \quad |\, x_1^1 \,| < \sqrt{2N} c_1 \tag{16}$$

in the limit $N \to \infty$. Thus the walker density always has a semicircle shape (Wigner's semicircle law), while the width of the semicircle is dependent on time and scaled by c_1. This result suggests that, after introducing a new rescaled variables $\lambda_j^m = x_j^m/c_m$, the vicious walk model in the diffusion scaling limit becomes equivalent to the matrix Brownian motion model [23–27] which is normalized so that the width of the semicircle is a constant.

The asymptotic forms of the dynamical correlation functions of matrix Brownian motion models were evaluated by Forrester, Nagao and Honner [28]. In particular, in the edge region of the semicircle, dynamical asymptotic correlations are described by the Airy function $\mathrm{Ai}(x)$. We can directly reinterpret Forrester, Nagao and Honner's result in the context of vicious walk model. Let us introduce rescaled temporal and spatial variables τ_m and X_j^m as

$$\begin{aligned} t_m &= \left(1 - \frac{\tau_m}{N^{1/3}}\right) T, \\ x_j^m &= c_m \left(\sqrt{2N} + \frac{X_j^m}{2^{1/2}N^{1/6}}\right) \end{aligned} \tag{17}$$

and take the limit $N \to \infty$. With an appropriate normalization, the dynamical correlation functions asymptotically have the same forms as in eq. (14) except that the quaternion matrices $B_{jl}^{\mu\nu}$ are replaced by

$$\mathcal{B}_{jl}^{\mu\nu} = \begin{bmatrix} \tilde{\mathcal{S}}_{jl}^{\mu\nu} & \tilde{\mathcal{I}}_{jl}^{\mu\nu} \\ \mathcal{D}_{jl}^{\mu\nu} & \tilde{\mathcal{S}}_{lj}^{\nu\mu} \end{bmatrix}, \quad j,l = 1,2,\cdots,N, \tag{18}$$

where

$$\mathcal{D}_{jl}^{\mu\nu} = \frac{1}{4}\left[\int_0^\infty dse^{-\tau_\mu s}\mathrm{Ai}(X_j^\mu+s)\frac{d}{ds}\{e^{-\tau_\nu s}\mathrm{Ai}(X_l^\nu+s)\} - \int_0^\infty dse^{-\tau_\nu s}\mathrm{Ai}(X_l^\nu+s)\frac{d}{ds}\{e^{-\tau_\mu s}\mathrm{Ai}(X_j^\mu+s)\}\right],$$

$$\tilde{\mathcal{I}}_{jl}^{\mu\nu} = \int_0^\infty dse^{-\tau_\nu s}\mathrm{Ai}(X_l^\nu-s)\int_s^\infty dve^{-\tau_\mu v}\mathrm{Ai}(X_j^\mu-v) - \int_0^\infty dse^{-\tau_\mu s}\mathrm{Ai}(X_j^\mu-s)\int_s^\infty dve^{-\tau_\nu v}\mathrm{Ai}(X_l^\nu-v)$$

and

$$\tilde{\mathcal{S}}_{jl}^{\mu\nu} = \begin{cases} \mathcal{S}_{jl}^{\mu\nu}, & \mu\ge\nu, \\ \mathcal{S}_{jl}^{\mu\nu}-\mathcal{G}_{jl}^{\mu\nu}, & \mu<\nu \end{cases}$$

with

$$\mathcal{S}_{jl}^{\mu\nu} = \int_0^\infty dse^{(\tau_\mu-\tau_\nu)s}\mathrm{Ai}(X_j^\mu+s)\mathrm{Ai}(X_l^\nu+s) + \frac{1}{2}\mathrm{Ai}(X_l^\nu)\int_0^\infty dse^{-\tau_\mu s}\mathrm{Ai}(X_j^\mu-s),$$

$$\mathcal{G}_{jl}^{\mu\nu} = \int_{-\infty}^\infty dse^{(\tau_\mu-\tau_\nu)s}\mathrm{Ai}(X_j^\mu+s)\mathrm{Ai}(X_l^\nu+s).$$

From the above asymptotic result we can extract information on limiting behavior. Let us firstly take the limit $\tau_\mu\to\infty$ with the time differences $\tau_\mu-\tau_\nu$ fixed. It can be readily seen that this limiting procedure is equivalent to set $\mathcal{D}_{jl}^{\mu\nu}$, $\tilde{\mathcal{I}}_{jl}^{\mu\nu}$ and the second term of $\mathcal{S}_{jl}^{\mu\nu}$ zeros. Then the quaternion determinant is reduced to an ordinary determinant and the asymptotic correlation functions become temporally homogeneous. We find that they are the dynamical correlation functions within the universality class of GUE. The equal time correlation functions are described by the Airy kernel [29,30] in the temporally homogeneous region. Because of the rescaling (17), it can be seen that the GUE universality class survives until time t very close to T: only when $T-t\sim O(N^{-1/3})$, the transition to GOE class occurs. Therefore we can conclude that the transition from GUE to GOE class is discontinuous in the limit $N\to\infty$.

The second interesting case is $X_j^\mu\to-\infty$ with the position differences $X_j^\mu-X_l^\nu$ fixed. This reproduces the asymptotic dynamical correlation functions in the bulk region. They are spatially homogeneous and the equal time correlations are equivalent to Pandey and Mehta's asymptotic result [18]. In this bulk region we can also observe the discontinuous transition from GUE to GOE.

In summary, nonintersecting Brownian motion of N particles in finite time interval $0\le t\le T$ was studied in one dimension. Dynamical correlation functions among many particles at many times were written in the forms of quaternion determinants. The asymptotic forms of the dynamical correlation functions in the limit $N\to\infty$ were evaluated and the universality class transition from GUE to GOE was found to be discontinuous. The compact asymptotic formula was derived above only after taking the diffusion scaling limit in which much information contained in the original lattice model was suppressed. In order to fully understand the lattice vicious walk model, we need to study other scaling limits as well. Further studies in this direction should be made in future works.

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